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Group actions on categories and equivariantization

Let $\mathcal{C}$ be a category,

consider $\underline{\mathrm{Aut}(\mathcal{C})}$ — obj: autoequiv. of $\mathcal{C}$; morphism: iso of functor

It's a monoidal sub cat. of the monoidal cat. $\mathrm{End}(\mathcal{C})$

If $\mathcal{C}$ is a monoidal cat.

we consider $\boxed{\mathrm{Aut}_{\otimes}(\mathcal{C})}$ of monoidal autoequiv. of $\mathcal{C}$

For a grp G,

let $\mathrm{Cat}(G)$ denote the monoidal cat.

whose obj. are elements of G,

the only morphisms are identities

$\otimes$ is $\cdot$, $\underline{\mathrm{Cat}(G) = \mathcal{C}_G(1)}$

Def 2.7.1

Let G be a group

(i) An action of G on a cat. $\mathcal{C}$ is a monoidal functor

$T: \underline{\mathrm{Cat}(G)} \to \underline{\mathrm{Aut}(\mathcal{C})}$

(ii) An action of G on a monoidal category $\mathcal{C}$ is a monoidal functor

$T: \mathrm{Cat}(G) \to \underline{\mathrm{Aut}_{\otimes}(\mathcal{C})}$

In these situations we also say that G acts on $\mathcal{C}$,

Let G be a grp acting on a cat. $\mathcal{C}$

$g \mapsto T_g$

Let $\gamma_{g,h}$ be the iso —

$T_g \circ T_h \simeq T_{gh}$ that defines the monoidal structure.

Def 2.7.2

A G-equivariant obj. in $\mathcal{C}$ is a pair (X, u) consisting of an object X of $\mathcal{C}$ and a family of iso

$u = \{u_g : T_g(X) \xrightarrow{\sim} X \mid g \in G\}$

s.t. the diagram commute

$$\begin{array}{ccc} T_g(T_h(X)) & \xrightarrow{T_g(u_h)} & T_g(X) \\ \gamma_{g,h}(X)\downarrow & \circlearrowright & \downarrow u_g \\ T_{gh}(X) & \xrightarrow{u_{gh}} & X \end{array}$$

One defines morphisms of equivariant obj. to be morphisms in $\mathcal{C}$ commuting with u_g, $g \in G$.

The cat. of G-equivariant obj of $\mathcal{C}$ or the G-equivariantization of $\mathcal{C}$ will be denoted by $\mathcal{C}^G$.

There is an obvious forgetful functor

$\mathrm{Forg}: \mathcal{C}^G \to \mathcal{C}$

A similar def. can be made for monoidal cat. $\mathcal{C}$, replacing $\mathrm{Aut}(\mathcal{C})$ by $\mathrm{Aut}_{\otimes}(\mathcal{C})$,

when $\mathcal{C}$ is a monoidal cat,

$\mathcal{C}^G$ is also a monoidal cat.

(X, u) $u_g: T_g(X) \xrightarrow{\sim} X$

(Y, v) $v_g: T_g(Y) \xrightarrow{\sim} Y$

$\underline{(X \otimes Y,\ w)}$

$w_g: T_g(X \otimes Y) \to X \otimes Y$

$T_g(X \otimes Y) \xrightarrow{J^{-1}} T_g(X) \otimes T_g(Y) \xrightarrow{u \otimes v} X \otimes Y$

Exercise 2.7.3

Show that the actions of a group G on $\underline{\mathrm{Vec}}$ (fid v.s.) viewed as an abelian category correspond to the elements of $H^2(G, \mathbb{k}^\times)$ while any action of G on Vec viewed as a monoidal cat. is trivial.

Proof: Let $(T, \gamma): \mathrm{Cat}(G) \to \mathrm{Aut}(\mathrm{Vec})$

一、$\gamma_{g,h}$ 由 $\mathbb{k}^\times$ 中元素决定

二、对应于 $H^2(G, \mathbb{k}^\times)$

一: for 1-dim v.s $\mathbb{k}$

$\underline{T_g \circ T_h(\mathbb{k})} \xrightarrow{(\gamma_{g,h})_{\mathbb{k}}} T_{gh}(\mathbb{k})$

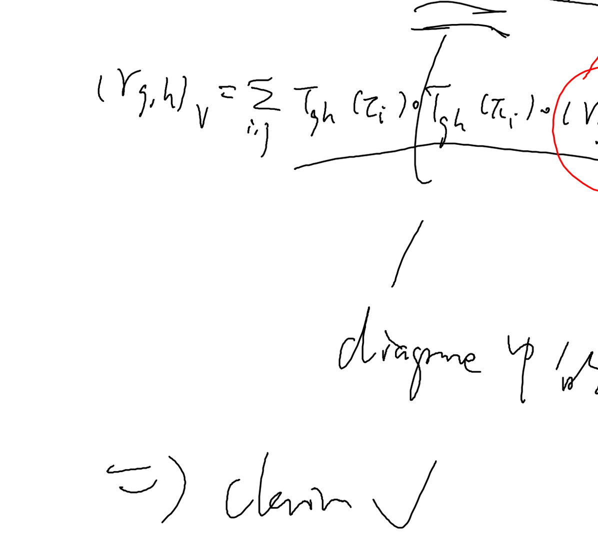
1 dim   dim

$\underline{(\gamma_{g,h})_{\mathbb{k}}} \leftrightarrow \mathbb{k}^\times$ $\underline{(\gamma_{g,h})_V}$

for $V \in \mathrm{Vec}$, $V \simeq (\mathbb{k})^{\oplus n}$

consider τ_i, π_i, $\tau_i: \mathbb{k} \to V$, $\pi_i: V \to \mathbb{k}$

By naturality, we have



Claim: By the diagram,

$$\boxed{(\gamma_{g,h})_V = \sum_{j=1}^{n} T_{gh}(\tau_j) \circ (\gamma_{g,h})_{\mathbb{k}} \circ (T_g T_h(\pi_j))}$$

$(\gamma_{g,h})_V = \sum_{i,j} T_{gh}(\tau_i) \circ \left(T_{gh}(\pi_i) \circ (\gamma_{g,h})_V \circ (T_g T_h)(\tau_j) \right) \circ T_g T_h(\pi_j)$

diagram 中 0/0

$\Rightarrow$ claim ✓

二: Since $T: \mathrm{Cat}(G) \to \mathrm{Aut}(\mathcal{C})$ is a monoidal functor,

$$\begin{array}{ccc} & T_g \circ T_h \circ T_k & \\ \gamma_{g,h} \otimes \mathrm{id} \swarrow & \circlearrowright & \searrow \mathrm{id} \otimes \gamma_{h,k} \\ T_{gh} \circ T_k & & T_g \circ T_{hk} \\ \gamma_{gh,k} \searrow & & \swarrow \gamma_{g,hk} \\ & T_{ghk} & \end{array}$$

$\mathbb{k}^\times$

$H^2(G, \mathbb{k}^\times)$ $Z^2(G, \mathbb{k}^\times)$

$\gamma_{gh,k} \cdot \gamma_{g,h} = \gamma_{g,hk} \cdot \gamma_{h,k}$

$\gamma: G \times G \to \mathbb{k}^\times$

$(T, \gamma) \overset{\eta}{\cong} (T', \gamma')$

$$\begin{array}{ccc} T_g \circ T_h(\mathbb{k}) & \xrightarrow{(\gamma_{g,h})_{\mathbb{k}}} & T_{gh}(\mathbb{k}) \\ (\eta_g \circ \eta_h)_{\mathbb{k}} \downarrow & \circlearrowright & \downarrow (\eta_{gh})_{\mathbb{k}} \\ T'_g \circ T'_h(\mathbb{k}) & \xrightarrow{(\gamma'_{g,h})_{\mathbb{k}}} & T'_{gh}(\mathbb{k}) \end{array}$$

$\Rightarrow (\gamma'_{g,h})_{\mathbb{k}} \cdot (\gamma_{g,h})^{-1}_{\mathbb{k}}$

$= \underline{(\eta_{gh})_{\mathbb{k}} (\eta_g)^{-1}_{\mathbb{k}} (\eta_k)^{-1}_{\mathbb{k}}} \in B^2$

$d_2(\mu)$

$\Rightarrow$ correspond to $H^2(G, \mathbb{k}^\times)$

(T_g, J_g) trivial

$F: \mathrm{Vec} \to \mathrm{Vec}$

monoidal autoequiv.

$\Leftrightarrow F: \mathbb{N} \to \mathbb{N}$

bijective 且 保持 数的 乘法

$F(0) = 0$, $\underline{F(1) = 1}$

$\Rightarrow$ $F = \mathrm{id}$

$\mathbb{N} \to \mathbb{N}$

$\dim V \leq \dim W$ $V \xrightarrow{f} W \xrightarrow{g} V$

证 $\dim F(V) \leq \dim F(W)$ $\underline{gf = \mathrm{id}}$

$F(V) \xrightarrow{F(f)} F(W)$

$\Rightarrow F(f)$ mono

$$\begin{array}{ccc} F(\mathbb{k}) & \xrightarrow{\text{同构}} & \mathbb{k} \\ \downarrow F(\theta) & & \downarrow \theta \\ F(\mathbb{k}x) & \xrightarrow{\quad} & \mathbb{k}x \end{array}$$